Three-Charge Black Holes and ¼ BPS States in Little String Theory

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Three-Charge Black Holes

BPS BH of charges (k,P,W) have been studied extensively in the last 20 years

- BH entropy : from Bekenstein-Hawking entropy formula,

$$S_{\mathrm{BH}} = \frac{A}{4G_N} = 2\pi\sqrt{kPW}$$

- Microscopic description of BH entropy: W **D1**-k **D5-**P on $T^4 \times S^1$ [Strominger,Vafa] [Maldacena,Moore,Strominger]

- Fuzzball program: attempt to describe such microstates by (classical) smooth geometries without horizon [Mathur][Bena,Warner]…

Such three-charge BHs can be also described as certain BHs in the near-horizon geometry of the fivebranes, which carry the charges (P,W) [Giveon,Kutasov,Rabinovici,Sever]

CHS CFT and LST

Near-Horizon Geometry of k NS5: 16 (space-time) SUSY

- Metric, Dilaton and H-flux

$$\begin{cases} ds^2 = dx^\mu dx_\mu + d\phi^2 + \underbrace{2k d\Omega_3^2}_{S^3} \\ g_s^2 = e^{-Q\phi} \\ \text{Constant H-flux over } S^3 \end{cases} \qquad Q = \sqrt{\frac{2}{k}}$$

- Exactly solvable world-sheet CFT, known as **C**allan-**H**arvey-**S**trominger CFT [Callan,Harvey,Strominger]

$$\mathbb{R}^{4,1} \times S^1 \times \underbrace{\underbrace{\mathbb{R}_\phi}_{\text{linear dilaton}} \times \underbrace{SU(2)_k}_{\text{SU(2) WZW at level k}}}$$

$$c = c_\phi + c_{\mathrm{SU(2)}} + c_{\mathrm{f}} = \left(1 + 3Q^2\right) + \left(3 - \frac{6}{k}\right) + 2 = 6$$

- $\mathbb{R}_\phi \times SU(2)_k$ preserves N=4 world-sheet superconformal symmetry

CHS CFT and LST

- Holographic dual to **Little String Theory** on the boundary $\phi \to \infty$ of the background where string coupling goes to zero.

Little String Theory [Aharony,Berkooz,Kutasov,Seiberg]

- The theory on NS5-branes

- Strongly coupled non-local theory that preserves 16 supercharges in 6D

- Decoupled from the bulk d.o.f. in the limit $g_s \to 0$ and $E \sim m_s$, different to the decoupling limit of AdS/CFT correspondence

CHS CFT and LST

Dabholkar-Harvey States in LST

- We are interested in ¼ BPS string states that carry momentum **P** and winding **W** around the circle S^1 of radius R

$$N_L = PW \qquad N_R = 0$$

$$M = \left| \frac{P}{R} + \frac{WR}{\alpha'} \right|$$

- However, CHS CFT is not useful because string coupling diverges as $\phi \to -\infty$, deep in the infinite throat of fivebranes

Double Scaled Little String Theory

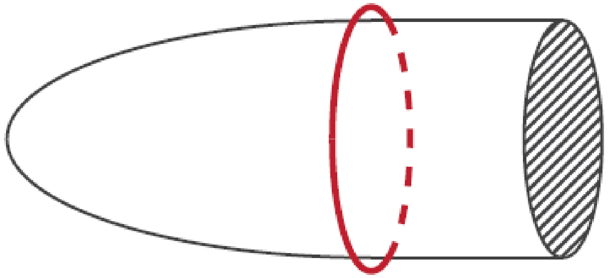
How to avoid the strong coupling? : Separate NS5-branes! [Giveon,Kutasov]

- For a single NS5-brane, fundamental string cannot see a CHS throat, because N=1 SU(2) WZW at k=1 contains a bosonic SU(2) current at level “-1”

- In the near-horizon geometry of k separated fivebranes (Coulomb branch), the throat is capped off, and the string coupling is bounded from above.

$$g_s(\mathrm{tip}) \simeq \frac{m_s}{\underbrace{M_W}_{\text{Mass of D1 between NS5s (type IIB)}}}$$

Mass of D1
between NS5s
(type IIB)



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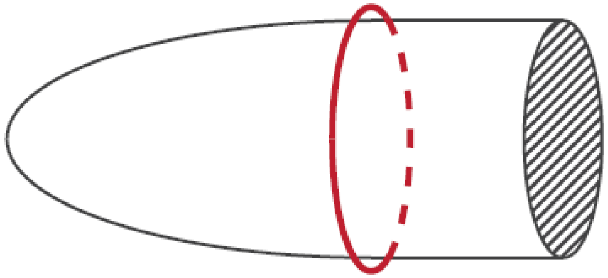


- As long as $m_s \ll M_W$, the effective string coupling remains weak

- This theory is called **Double Scaled Little String Theory**

¼ BPS States in DSLST

What We Want to Study Today

- 1/4 BPS states with two charges (P,W), Dabholkar-Harvey states, in the near-horizon geometry of k separated NS5-branes, for which

$$N_L = PW \qquad N_R = 0 \qquad M = \left| \frac{P}{R} + \frac{WR}{\alpha'} \right|$$

- High-energy entropy of BPS states

- THEIR ENTROPY can be read off from the ELLIPTIC GENUS that counts the right-moving ground states

¼ BPS States in DSLST

Why ¼ BPS States in DSLST ?

- Black Hole Microstate Geometry: fivebrane charges are **separated**…

e.g. two-charge states in flat 10D [Lunin,Mathur][Lunin,Maldacena,Maoz]…

$$ds^2 = f_1^{-1/2} f_5^{-1/2} \left[-(dt - A)^2 + (dy + B)^2 \right] + f_1^{1/2} f_5^{1/2} d\vec{x} \cdot d\vec{x} + f_1^{1/2} f_5^{-1/2} d\vec{z} \cdot d\vec{z}$$

$$f_5 = 1 + \frac{k}{L} \boxed{\int_0^L \frac{dv}{|\vec{x} - \vec{F}(v)|^2}}\,, \qquad f_1 = 1 + \frac{k}{L} \int_0^L \frac{|\dot{\vec{F}}(v)|^2 dv}{|\vec{x} - \vec{F}(v)|^2}$$

- Microstate geometries have an entropy of corresponding BH?

- Let’s address the above question in the context of LST!

¼ BPS States in DSLST

Naively, we expected that

Entropy of DH states in DSLST	$\stackrel{?}{=}$	Entropy of Three-Charge BH

This is because

- Elliptic genus is independent of the positions of NS5s:
  However, the N=4 character decomposition will partly supports this claim

- Mass of DH states is independent of the position of NS5s

- The physics of high mass states, DH state with large (P,W), can be hardly affected by small W-boson mass of fivebranes separated infinitesimally

String-BH Transition in LST

String-Black Hole Transition

- However we will see that the spectrum is qualitatively different when NS5-branes are separated by any finite distance, and when they are coincident

- String-BH transition in the phase space of LST

- We will discuss why this can happen, and comment on physical implications

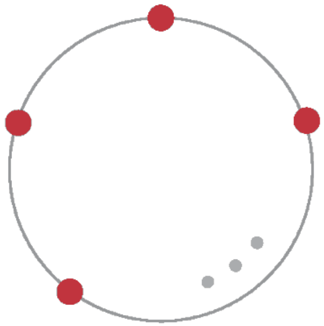
Exactly Solvable CFT for DSLST

Exactly Solvable CFT:

- We will argue that, unless NS5 branes are coincident, the elliptic genus of DSLST is still independent of the fivebrane positions

- When NS5-branes are evenly placed on a circle of radius R_0, CHS CFT is deformed to another solvable CFT

$$\mathbb{R}_\phi \times \underbrace{\left(U(1)_k \times \frac{SU(2)_k}{U(1)}\right)/\mathbb{Z}_k}_{\text{SU(2)}_k\text{ WZW}} \to \left(\underbrace{\frac{SL(2,\mathbb{R})_k}{U(1)}}_{\text{Cigar}} \times \underbrace{\frac{SU(2)_k}{U(1)}}_{\text{Minimal model}}\right)/\mathbb{Z}_k$$

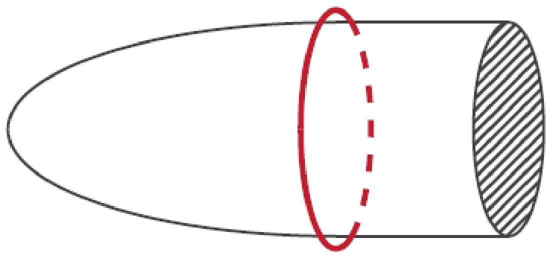


k

- Although each of coset CFTs has N=2 SCA, DSLST has N=4 SCA

N=2 Cigar CFT and Minimal Model

Cigar CFT : $\frac{SL(2,\mathbb{R})_k}{U(1)}$ **(**Kazama-Suzuki Construction)



- N=(2,2) Non-linear sigma model on a target space
  with metric $ds^2 = k\left(dr^2 + \tanh^2 r d\theta^2\right)$

- Non-trivial dilaton profile: N=2 SCFT (beta function vanishes)

$$\Phi = \Phi_0 - 2\log\cosh r$$

- Central charge: $c_{\rm cig} = 3\left(1 + \frac{2}{k}\right)$

Minimal Model : $\frac{SU(2)_k}{U(1)}$

- N=(2,2) Landau-Ginzburg Model: $W = X^{k+2} + Y^2 + Z^2$

- Central charge: $c_{\rm min} = 3\left(1 - \frac{2}{k}\right)$

How to define and compute the elliptic genus of DSLST?

Elliptic Genus of Compact CFTs

Definition : For N=(0,2) SUSY theories,

$$\mathcal{E}(\tau,\vec{z}) = \mathrm{Tr}_{\mathcal{H}_{\rm RR}}\left[(-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} e^{2\pi i \vec{z}\cdot\vec{J}}\right] \qquad q = e^{2\pi i\tau}$$

- H_{RR}: Hilbert space of SCFT in Ramond-Ramond sector
- J: Global charges that commute with (0,2) supercharges collectively
- Witten Index: $\mathcal{I}(\tau) = \mathcal{E}(\tau, z=0)$

Properties

- Modular and elliptic : Jacobi form of weight 0 and index n

$$\mathcal{E}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = e^{\frac{2\pi i n c z^2}{c\tau+d}}\mathcal{E}(\tau,z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\mathcal{E}(\tau, z+k\tau+l) = e^{-2\pi i n(k^2\tau+2kz)}\mathcal{E}(\tau,z) \qquad k,l\in\mathbb{Z}$$

Elliptic Genus of Compact CFTs

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- H_{RR}: Hilbert space of SCFT in Ramond-Ramond sector

- J: Global charges that commutes with (0,2) supercharges collectively

Properties

- Modular and elliptic : Jacobi form of weight 0 and index n

- Holomorphic in terms of q : only discrete states with $\bar{L}_0 \doteq 0$ can contribute
(right-moving ground states)

Elliptic Genus of DSLST

Elliptic Genus of DSLST

$$\mathcal{E}_{\mathrm{DSLST}} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\left[(-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} e^{2\pi i z(2J^3_{\mathrm{R}})_0}\right]$$

- $\mathbf{H_{RR}}$: Hilbert space of N=4 SCFT with c=6 in Ramond-Ramond sector

$$\left(\frac{SL(2,\mathbb{R})_k}{U(1)} \times \frac{SU(2)_k}{U(1)}\right)/\mathbb{Z}_k$$

- $\mathbf{2J_R^3}$: $U(1)_R$ current of $SU(2)_R$ current in N=4 SCA,

$$2J^3_{\mathrm{R}} = J^{\mathrm{cig}}_{\mathrm{R}} + J^{\mathrm{min}}_{\mathrm{R}}$$

Elliptic Genus of DSLST

Z_k Orbifold

- Orbifold action is generated by $e^{2\pi i(2J^3_{\rm R})}$ [Kawai,Yamada,Yang]

$$\mathcal{E}_{\rm DSLST}(\tau,z) = \frac{1}{k}\sum_{\alpha,\beta=0}^{k-1} q^{\frac{c}{6}\alpha^2}\left(e^{2\pi i z}\right)^{\frac{c}{3}\alpha}\mathcal{E}_{\rm cig}(\tau, z+\alpha\tau+\beta)\mathcal{E}_{\rm min}(\tau, z+\alpha\tau+\beta)$$

Vacuum energy and U(1) R-charge shifts in twisted sectors

α for twisted sector and β for projection

Minimal Model: Using the localization technique, [Witten]

$$\mathcal{E}_{\rm min}(\tau,z) = \frac{\vartheta_1\left(\tau, \left(1-\frac{1}{k}\right)z\right)}{\vartheta_1\left(\tau, \frac{1}{k}z\right)}$$

Odd Jacobi theta function: $\vartheta_1(\tau,z) = -iq^{1/8}e^{\pi i z}\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi i z}q^n)(1-e^{-2\pi i z}q^{n-1})$

Elliptic Genus of Cigar

Needs more elaboration to obtain the elliptic genus of cigar CFT

- Non-compact CFTs have both discrete and continuum states

Perform the localization in a GLSM that flows in the infrared to cigar CFT [Hori,Kapustin]

$$\mathcal{E}_{\rm cig}(\tau,z) = \sqrt{k\tau_2}\int_0^1 ds_1 \int_0^1 ds_2 \sum_{(p,w)\in\mathbb{Z}^2} \frac{\vartheta_1\big(\tau, s_1\tau+s_2+z(1+1/k)\big)}{\vartheta_1\big(\tau, s_1\tau+s_2+z\big)} \times e^{-2\pi i z w} e^{-2\pi i s_2 p} q^{l_0} \bar{q}^{\bar{l}_0}$$

$$l_0 = \frac{1}{4k}\big(p-k(w+u_2)\big)^2 \,, \qquad \bar{l}_0 = \frac{1}{4k}\big(p+k(w+u_2)\big)^2$$

[Eguchi,Sugawara][Ashok,Troost][Murthy][Ashok,Doroud,Troost] …

- Modular and Elliptic, but Non-holomorphic !

Separation of Discrete States

We are interested in discrete spectrum of the theory.

Need to separate the contribution from scattering states.

Scattering state contributions arise from the spectral asymmetry,

$$\rho_B(E) - \rho_F(E) \neq 0$$

Densities of states are related to scattering phase shifts, which is very difficult to compute in general

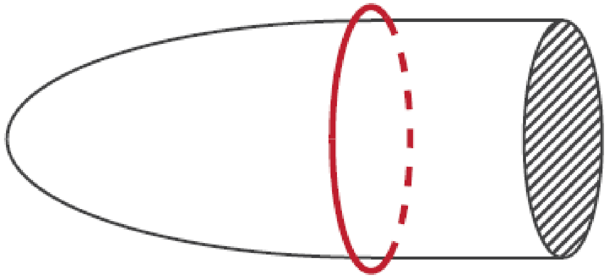
However, their difference can be determined exactly by asymptotic data, thank s to SUSY! [Akhoury,Comet]

Separation of Discrete States

The oscillatory modes of the string do not affect the spectrum asymmetry

Reduction to Quantum Mechanics:

$$ds^2 = k\left(dr^2 + \tanh^2 r\, d\theta^2\right)$$



- Scherk-Schwarz reduction:

$$\partial_1 r = 0$$

$$\partial_1 \theta = w$$

- Bosonic Lagrangian: governs the dynamics of c.o.m. of the string that winds S^1

$$\mathcal{L}^B_{\mathrm{QM}} = \frac{k}{2}\left(\frac{dr}{dt}\right)^2 + \frac{k}{2}\tanh^2 r\left(\frac{d\theta}{dt}\right)^2 \underbrace{- \frac{k}{2}w^2\tanh^2 r}_{\text{string tension}}$$

Separation of Discrete States

Spectral Asymmetry

- Four degenerate states: two of them are bosons and the others are fermions

$$\{\bar{\psi}_+, \psi_+\} = \{\bar{\psi}_-, \psi_-\} = \frac{1}{k} \longrightarrow \left[\begin{array}{l} \langle x|B_1\rangle = f_1(r,\theta)|++\rangle \\ \langle x|F_1\rangle = g_1(r,\theta)|-+\rangle \\ \langle x|B_2\rangle = f_2(r,\theta)|--\rangle \\ \langle x|F_2\rangle = g_2(r,\theta)|+-\rangle \end{array} \right.$$

- SUSY provides strong constraints between phase-shift factors,

$$\left. \begin{array}{l} Q_+|B_1\rangle \sim |F_1\rangle \\ \\ Q_+|F_2\rangle \sim |B_2\rangle \end{array} \right] \longrightarrow \left[\begin{array}{l} \left[-\frac{i}{\sqrt{k}}\frac{\partial}{\partial r} - \frac{1}{\sqrt{k}}\frac{\partial}{\partial \theta} - i\sqrt{k}w\right] f_1(r,\theta) \propto g_1(r,\theta) \\ \\ \left[-\frac{i}{\sqrt{k}}\frac{\partial}{\partial r} - \frac{1}{\sqrt{k}}\frac{\partial}{\partial \theta} - i\sqrt{k}w\right] g_2(r,\theta) \propto f_2(r,\theta) \end{array} \right.$$

Separation of Discrete States

Spectral Asymmetry

- Scattering phase shifts: (K : radial momentum)

$$e^{2i\left(\delta_B^1(K)-\delta_F^1(K)\right)} = -\frac{K+i(p+kw)}{K-i(p+kw)}$$

$$e^{2i\left(\delta_B^2(K)-\delta_F^2(K)\right)} = -\frac{K-i(p+kw)}{K+i(p+kw)}$$

- Using the standard relation between the spectral density and the phase shift,

$$\rho_B(E)-\rho_F(E) = \frac{1}{\pi}\frac{\partial}{\partial E}\left(\delta_B(E)-\delta_F(E)\right)$$

one can obtain the spectral (a)symmetry

$$\rho_B^1(K)-\rho_F^1(K) = -\rho_B^2(K)+\rho_F^2(K) = \frac{1}{2\pi i}\left[\frac{1}{K+i(p+kw)}+\frac{1}{-K+i(p+kw)}\right]$$

Separation of Discrete States

Contribution from the Scattering States

$$\mathcal{E}_{\rm c} = \sum_{p,w} \int_0^\infty dK \left[\left(\rho_B^1(K) - \rho_F^1(K)\right) e^{-\pi i z} + \left(\rho_B^2(K) - \rho_F^2(K)\right) e^{+\pi i z} \right]$$

$$\times \left(e^{-2\pi\tau_2}\right)^{E(K,p,w)} \left(e^{2\pi i \tau_1}\right)^{P(p,w)} e^{2\pi i z J(p,w)} \times \underbrace{\prod_{n=1}^{\infty} \left[\frac{\left(1 - q^n e^{2\pi i z}\right)\left(1 - q^n e^{-2\pi i z}\right)}{\left(1 - q^n\right)^2} \right]}_{\text{Oscillator Modes}}$$

$$E(K,p,w) = \frac{K^2}{2k} + \frac{p^2}{2k} + \frac{kw^2}{2}$$

$$P(p,w) = -pw$$

$$J(p,w) = \frac{p}{k} - w$$

	$\vert B_1\rangle$	$\vert F_1\rangle$	$\vert B_2\rangle$	$\vert F_2\rangle$
$U(1)_l$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
$U(1)_r$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$

Separation of Discrete States

Discrete State Contributions:

$$\mathcal{E}_{\rm cig} = \mathcal{E}_{\rm d} + \mathcal{E}_{\rm c}$$

where the discrete and continuum parts are given by

$$\mathcal{E}_d = +\frac{i\vartheta_1(\tau,z)}{\eta(q)^3}\sum_{\alpha=0}^{k-1}\sum_{w\in\mathbb{Z}}\frac{q^{(-\alpha+kw)w}e^{2\pi i z(-\alpha/k+2w)}}{1-e^{2\pi i z}q^{-\alpha+kw}}$$

$$\mathcal{E}_{\rm c} = -\frac{\vartheta_1(\tau,z)}{2\eta(q)^3}\sum_{p,w\in\mathbb{Z}}\int_0^\infty dK\,\frac{1}{\pi}\left[\frac{1}{K+i(p+kw)}+\frac{1}{-K+i(p+kw)}\right](q\bar{q})^{\frac{K^2}{4k}}$$
$$\times\, q^{\frac{1}{4k}(p-kw)^2}\bar{q}^{\frac{1}{4k}(p+kw)^2}e^{2\pi i z(\frac{p}{k}-w)}$$

Mock Modular Form

Contribution to elliptic genus from scattering states of non-compact CFTs can be always expressed in terms of the non-holomorphic Eichler integrals $R_{k,l}$

For instance, in the case of cigar CFT,

$$\mathcal{E}_{\rm c} = \frac{i\vartheta_1(\tau,z)}{\eta(q)^3}\cdot\frac{1}{k}\sum_{\beta,\gamma=1}^{k} e^{2\pi i\frac{\beta\gamma}{k}} q^{\frac{\beta^2}{k}}\left(e^{2\pi i\frac{z}{k}}\right)^{2\beta}\cdot\left(-\frac{1}{2}\sum_{l=1}^{2k} R^{+}_{k,l}(\tau)\vartheta_{k,l}\left(\tau,\frac{z+\beta\tau+\gamma}{k}\right)\right)$$

$$R^{\pm}_{k,l}(\tau) = \sum_{\lambda=l+2k\mathbb{Z}} \mathrm{sgn}(\lambda\pm\epsilon)\mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{k}}|\lambda|\right) q^{-\frac{\lambda^2}{4k}}$$

$$\vartheta_{k,l}(\tau,z) = \sum_{\lambda=l+2k\mathbb{Z}} q^{\lambda^2/4k} e^{2\pi i z\lambda}$$

The Eichler integral often appears, in math, to make a holomorphic mock modular form a non-holomorphic modular form

Mock Modular Form

Indeed, the discrete part can be rewritten as

$$\mathcal{E}_d = +\frac{i\vartheta_1(\tau,z)}{k\eta(q)^3}\sum_{\beta,\gamma=1}^{k} e^{2\pi i\frac{\beta\gamma}{k}} q^{\frac{\beta^2}{k}} \left(e^{2\pi i z}\right)^{\frac{2\beta}{k}} \mathcal{A}_{1,k}(\tau, \frac{z+\beta\tau+\gamma}{k})$$

where $A_{1,k}$, the Appell-Lerch sum, is a well-known mock modular form

$$\mathcal{A}_{1,k}(\tau,z) = \sum_{t\in\mathbb{Z}} \frac{q^{kt^2}\left(e^{2\pi i z}\right)^{2kt}}{1-\left(e^{2\pi i z}\right)q^t}$$

Full elliptic genus is of course modular, but non-holomorphic!

$$\mathcal{E}_{\rm cig} = \mathcal{E}_d + \mathcal{E}_c = +\frac{i\vartheta_1(\tau,z)}{k\eta(q)^3}\sum_{\beta,\gamma=1}^{k} e^{2\pi i\frac{\beta\gamma}{k}} q^{\frac{\beta^2}{k}} \left(e^{2\pi i z}\right)^{\frac{2\beta}{k}} \hat{\mathcal{A}}_{1,k}(\tau, \frac{z+\beta\tau+\gamma}{k})$$

$$\hat{\mathcal{A}}_{1,k}(\tau,z) = \mathcal{A}_{1,k}(\tau,z) - \frac{1}{2}\sum_{l=1}^{2k} R^{+}_{k,l}(\tau)\vartheta_{k,l}(\tau,z)$$

In summary, the discrete part of elliptic genus of DSLST is

$$\mathcal{E}^{d}_{\mathrm{DSLST}}(\tau,z)=\frac{1}{k}\sum_{\alpha,\beta=0}^{k-1} q^{\frac{c}{6}\alpha^2}\left(e^{2\pi i z}\right)^{\frac{c}{3}\alpha}\mathcal{E}^{d}_{\mathrm{cig}}(\tau,z+\alpha\tau+\beta)\mathcal{E}_{\mathrm{min}}(\tau,z+\alpha\tau+\beta)$$

where

$$\mathcal{E}^{d}_{\mathrm{cig}}(\tau,z)=\frac{i\vartheta_1(\tau,z)}{\eta(q)^3}\sum_{\alpha=0}^{k-1}\sum_{w\in\mathbb{Z}}\frac{q^{(-\alpha+kw)w}e^{2\pi i z(-\alpha/k+2w)}}{1-e^{2\pi i z}q^{-\alpha+kw}}$$

$$\mathcal{E}_{\mathrm{min}}(\tau,z)=\frac{\vartheta_1\left(\tau,\left(1-\frac{1}{k}\right)z\right)}{\vartheta_1\left(\tau,\frac{1}{k}z\right)}$$

Discuss various properties of the above result!

N=4 Character Decomposition

N=4 SUSY: CFT of our interest has an N=4 superconformal algebra with c=6

$$\left(\frac{SL(2,\mathbb{R})_k}{U(1)} \times \frac{SU(2)_k}{U(1)}\right)/\mathbb{Z}_k$$

, although each of coset CFTs has N=2 superconformal algebra with

$$c_{\mathrm{cig}} = 3\left(1 + \frac{2}{k}\right) \quad \text{and} \quad c_{\mathrm{min}} = 3\left(1 - \frac{2}{k}\right)$$

NB Two currents of $SU(2)_R$ in addition to $U(1)_R$ current are in the twisted sector

Discrete States of the CFT must be in a representation of N=4 SCA

N=4 Character Decomposition

Elliptic genus must be decomposed into a sum of N=4 Ramond characters

$$\mathrm{ch}^{(m)}_{h,j}(\tau, z) = \mathrm{Tr}_{V^{(m)}_{h,j}}\left[(-1)^F e^{2\pi i z (2J^3_{\mathrm{R}})_0} q^{L_0 - c/24}\right]$$

$V^{(m)}_{h,j}$: irreducible highest weight rep. of N=4 SCA with c=6(m-1)

- N=4 Massless Characters with **c=6** [Eguchi,Taormina]

$$\mathrm{ch}^{(2)}_{\frac{1}{4},1}(t, z) = i\frac{\vartheta_{11}(\tau, z)^2}{\eta(q)^3\vartheta_{11}(\tau, 2z)} \cdot \sum_{k\in\mathbb{Z}} q^{mk^2}\xi^{2mk} \sum_{a=-1}^{2} \frac{(\xi q^k)^a}{1-\xi q^k} \qquad \xi = e^{2\pi i z}$$

$$\mathrm{ch}^{(2)}_{\frac{1}{4},0}(t, z) = i\frac{\vartheta_{11}(\tau, z)^2}{\eta(q)^3\vartheta_{11}(\tau, 2z)} \cdot (-1)\sum_{k\in\mathbb{Z}} q^{mk^2}\xi^{2mk} \sum_{a=0}^{1} \frac{(\xi q^k)^a}{1-\xi q^k}$$

- N=4 Massive Characters with **c=6**

$$\mathrm{ch}^{(2)}_{h,1}(t, z) = i\frac{\vartheta_{11}(\tau, z)^2}{\eta(q)^3\vartheta_{11}(\tau, 2z)} \cdot q^{h-\frac{3}{8}}\Big(\vartheta_{2,-1}(\tau, z) - \vartheta_{2,1}(\tau, z)\Big)$$

N=4 Character Decomposition

N=4 Character Decomposition of *E*$_{\text{DSLST}}$

$$\mathcal{E}^{d}_{\text{DSLST}}(\tau,z) = (k-1)\mathrm{ch}^{(2)}_{\frac{1}{4},0}(\tau,z) + \sum_{n=1}^{\infty} a_n \mathrm{ch}^{(2)}_{\frac{1}{4}+n,1}(\tau,z)$$

where the coefficients an are given by

$$-\frac{1}{2}\eta(\tau)^3 \sum_{n=1} a_n q^{n-\frac{1}{8}} = \mathcal{F}^{k,1}_2(q) - (k-1)\mathcal{F}^{2,1}_2(q)$$

$$\mathcal{F}^{k,1}_2 = \left[\sum_{\substack{r,s\in\mathbb{Z} \\ 0<s<kr}} - k \sum_{\substack{r,s\in\mathbb{Z} \\ 0<ks<r}} \right] s q^{rs}$$

For instance, first few coefficients are

$$a_1 = 2k-4 \ , \qquad a_2 = 8k-20 \ , \qquad a_3 = \begin{cases} 6 & \text{if} \quad k=3 \\ 22k-66 & \text{if} \quad k>3 \end{cases}$$

What Are These States?

What the Decomposition implies …

- Vertex operators corresponding to massless character $\left(j = 0, \frac{1}{2}, .., \frac{k-2}{2}\right)$

$$\mathcal{O}^{(0)}_{j,0} \equiv V^{\text{susy}}_{j;j+1,j+1}\left(-\frac{1}{2}, -\frac{1}{2}\right) \cdot \tilde{V}^{\text{susy}}_{j;j,j}\left(+\frac{1}{2}, +\frac{1}{2}\right)$$

$V^{\text{susy}}_{j;m,\bar{m}}(\alpha, \bar{\alpha})$: primary operators of cigar CFT with
$$h = \frac{(m+\alpha)^2 - j(j+1)}{k} + \frac{\alpha^2}{2}$$
$$r = \frac{2(m+\alpha)}{k} + \alpha$$

$\tilde{V}^{\text{susy}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta, \bar{\beta})$: primary operators of min. model with
$$h = \frac{\tilde{j}(\tilde{j}+1) - (\tilde{m}+\beta)^2}{k} + \frac{\beta^2}{2}$$
$$r = -\frac{2(\tilde{m}+\beta)}{k} + \beta$$

where α and β are spectral-flow parameters

What Are These States?

What the Decomposition implies …

$$\mathcal{O}^{(0)}_{j,0} \equiv V^{\rm susy}_{j;j+1,j+1}\Big(-\frac{1}{2},-\frac{1}{2}\Big)\cdot \tilde{V}^{\rm susy}_{j;j,j}\Big(+\frac{1}{2},+\frac{1}{2}\Big)$$

- These (k-1) operators are related to NS vertex operators via the spectral flow, which describe the position of the fivebranes.

- Why (k-1) rather than k?: The operator for the c.o.m. is non-normalizable [Callan,Harvey,Strominger]

For N=4 massless (as well as massive) representations in the decomposition, the independence of the position moduli is obvious unless some of NS5s collide

Finally, let's estimate the entropy of 1/4 BPS states in DSLST

Entropy of Dabholkar-Harvey States

How many ¼-BPS string states of P and W with PW=N_L ?

- Contributions from $R^{4,1}xS^1$ (or, RxT^4xS^1) are trivial

- Contributions from the exact CFT $\left(\frac{SL(2,\mathbb{R})_k}{U(1)} \times \frac{SU(2)_k}{U(1)}\right)/\mathbb{Z}_k$:

$$\mathcal{E}^d_{\rm DSLST}(\tau,z) = \sum_{N=0}^{\infty} D(N,z) q^N$$

$$q = e^{2\pi i\tau}$$

$$D(N,z) = \oint \frac{dq}{2\pi i} \frac{\mathcal{E}^d_{\rm DSLST}(\tau,z)}{q^{N+1}}$$

At z=1/2, the elliptic genus essentially becomes the partition function

For large N, use the saddle-point approximation to evaluate the integral at $q = 1-\epsilon$ with $\epsilon \simeq \sqrt{\frac{\pi^2}{N+1}\left(1-\frac{1}{k}\right)}$, then $D\left(N,\frac{1}{2}\right) \simeq e^{2\pi\sqrt{\left(1-\frac{1}{k}\right)N}}$

Entropy of Dabholkar-Harvey States

How many ¼-BPS string states of P and W with PW=N_L ?

- Collecting all the contributions, one obtains

$$S_{\text{string}} = 2\pi\sqrt{\left(1+1-\frac{1}{k}\right)N_L}$$

- It agrees with a **naïve** expectation from the Cardy formula

$$S = 2\pi\sqrt{\frac{c_{\text{eff}}}{6}N_L} + 2\pi\sqrt{\frac{\bar{c}_{\text{eff}}}{6}N_R}$$

$$c_{\text{eff}} = c - 24\Delta_{\text{min}} = 6\left(2-\frac{1}{k}\right)$$

- Much smaller than the entropy of the corresponding three-charge BH

$$S_{\text{BH}} = 2\pi\sqrt{k\cdot PW}$$

What are the implications of the result?

Two-Charge BH in LST

When fivebranes are coincident, there is another candidate for a state with the same quantum numbers as DH states, which is two-charge BH in LST

(Extremal) Two-Charge BH in LST [Giveon,Kutasov,Rabinovici,Sever]

$$ds^2 = -f(r)dt^2 + \frac{k}{2}\frac{dr^2}{f(r)r^2}\ , \qquad f(r) = \left(1 - \frac{M_{\text{BH}}}{r}\right)^2$$

$$\Phi = -\frac{1}{2}\log\left(\sqrt{\frac{k}{2}}r\right) \qquad A_t = \frac{1}{r}M_{\text{BH}} \qquad M_{\text{BH}} = \left|\frac{P}{R} + \frac{WR}{2}\right|$$

- Nothing but three-charge BH of Strominger and Vafa, except we are viewing it as a state in LST

Two-Charge BH in LST

(Extremal) Two-Charge BH in LST [Giveon,Kutasov,Rabinovici,Sever]

- Exactly solvable world-sheet CFT

$$\frac{SL(2,\mathbb{R})_k \times U(1)}{U(1)} \times SU(2) \times \mathbb{T}^4$$

[1] U(1) that is being gauged is a combination of CSA of SL(2,R) and U(1) of the circle, which depends on P and W and non-extremal parameter

[2] To have a finite-mass BH, the fivebrane world-volume R^4 has to be replaced by a compact space, say T^4. Otherwise 10D string coupling diverges.

- Entropy of two-charge BH in LST

$$S_{\rm BH} = 2\pi\sqrt{kPW}$$

(DS)LST on T^4

LST on T^4

- The theory on NS5 branes lives in 0+1 dimensions, i.e., quantum mechanics in which the vacuum is characterized by a wave-function over the moduli space

- What we discovered implies that LST on T^4 has non-trivial vacuum structure

e.g. [1] one set of vacua: quantization of moduli space of distinct NS5s

$$S = 2\pi\sqrt{\left(2 - \frac{1}{k}\right) PW}$$

[2] the other set of vacua: quantization of moduli space of coincident NS5s

$$S = 2\pi\sqrt{kPW}$$

(DS)LST on T^4

LST on T^4

- Sharp String-BH transition between different phases of LST

In the 2D N=(2,2) gauge theory describing the system of NS5s and F1s,

- Two branches: Higgs branch for F1s on top of NS5s, and Coulomb branch for F1s away from NS5s

- Two different CFTs with different central charges for two branches [Witten]

- Reminiscent of our result

Infinite throat of NS5s plays a key role in both of our and Witten's analysis

Horowitz-Polchinski Transition and LST

Horowitz-Polchinski String-BH Transition

- Start with a typically highly excited fundamental string state

- Slowly raise the string coupling constant

- At some point, the size of horizon of a BH with the same quantum numbers as the fundamental string exceeds the string scale

- Then, the fundamental string picture breaks down and the system is better described as a black hole

Horowitz-Polchinski Transition and LST

Non-extremal LST

- Entropy for non-extremal string states and BH states,

$$S_{\text{string}} = \sqrt{2}\pi \cdot \sqrt{2 - \frac{1}{k}} \cdot \left(\sqrt{M^2 - P_L^2} + \sqrt{M^2 - P_R^2} \right)$$
$$S_{\text{BH}} = \sqrt{2}\pi \cdot \sqrt{k} \cdot \left(\sqrt{M^2 - P_L^2} + \sqrt{M^2 - P_R^2} \right)$$

- Positions of fivebranes are no longer moduli. They attract each other and eventually collide.

Horowitz-Polchinski Transition and LST

When the non-extremal parameter is small and NS5s are well-separated initially so that the effective string coupling is small,

- For a long period, non-extremal string states dominate the entropy

$$S_{\text{string}} = \sqrt{2}\pi \cdot \sqrt{2 - \frac{1}{k}} \cdot \left(\sqrt{M^2 - P_L^2} + \sqrt{M^2 - P_R^2} \right)$$

- As time goes by, NS5-branes move toward each other and string coupling grows

- At late times, NS5-branes eventually collide and the system makes a transition to a BH phase

$$S_{\text{BH}} = \sqrt{2}\pi \cdot \sqrt{k} \cdot \left(\sqrt{M^2 - P_L^2} + \sqrt{M^2 - P_R^2} \right)$$

Horowitz-Polchinski string-BH transition happens dynamically and smoothly in the non-extremal case of LST

Fuzzball Program and LST

Fuzzball Program

- Attempt to find horizonless geometry that looks like BH outside the horizon, but deviate from BH near the would-be horizon

- Entropy of such smooth geometry = Entropy of BH (?)

A Subtlety with the Fuzzball Program

- There must exist smooth solutions in SUGRA that correspond to the ¼ BPS string states in the background of fivebranes

e.g. two-charge states in flat 10D [Lunin,Mathur][Lunin,Maldacena,Maoz]…

$$ds^2 = f_1^{-1/2} f_5^{-1/2} \big[-(dt - A)^2 + (dy + B)^2 \big] + f_1^{1/2} f_5^{1/2} d\vec{x} \cdot d\vec{x} + f_1^{1/2} f_5^{-1/2} d\vec{z} \cdot d\vec{z}$$

$$f_5 = 1 + \frac{k}{L} \int_0^L \frac{dv}{|\vec{x} - \vec{F}(v)|^2}\,, \qquad f_1 = 1 + \frac{k}{L} \int_0^L \frac{|\dot{\vec{F}}(v)|^2 dv}{|\vec{x} - \vec{F}(v)|^2}$$

Fuzzball Program and LST

A Subtlety with the Fuzzball Program

- Presumably, these geometries are the same as three-charge BH solutions at large distance from the horizon

- Are they microstates of three-charge BH ?

- No, we discovered that the fundamental string states in the separated fivebrane background and the BH states are qualitatively different!

- They live in different vacua of LST with different entropies

Our result suggests that a horizonless geometry that well approximates the BH cannot necessarily be thought of as a microstate of the BH

Summary and Outlook

¼ BPS states in (DS)LST

- Elliptic genus of DSLST

- Entropy of ¼ BPS states in DSLST is much less than that of three-charge BH

String-BH Transition: LST on T^4 has very non-trivial vacuum structure

- Some of them correspond to string states, and the other to BH states

- Implications to Horowitz-Polchinski transition, Fuzzball program, and so on

Unified description for Little String Theory that contains many branches, some of which corresponds to string states and the other to BH states ?

Dipole D-brane charges + F1 in Coulomb phase ?